Alexander Müller (s5193400)

1. (a)

Solution:

1. Axiom 1: $\alpha \to (\beta \to \alpha)$

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha}\,(w \Rightarrow)}{\alpha \Rightarrow \beta \to \alpha}\,(\Rightarrow\to)}{\Rightarrow \alpha \to (\beta \to \alpha)}\,(\Rightarrow\to)$$

2. Axiom 2: $(\alpha \to (\alpha \to \gamma)) \to (\alpha \to \gamma)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\gamma \Rightarrow \gamma \qquad \alpha \Rightarrow \alpha}{\alpha \to \gamma, \alpha \Rightarrow \gamma}\,(\to\Rightarrow)}{\alpha \to (\alpha \to \gamma), \alpha, \alpha \Rightarrow \gamma}\,(\to\Rightarrow)}{\alpha \to (\alpha \to \gamma), \alpha \Rightarrow \gamma}\,(c \Rightarrow)}{\alpha \to (\alpha \to \gamma) \Rightarrow \alpha \to \gamma}\,(\Rightarrow\to)}{\Rightarrow (\alpha \to (\alpha \to \gamma)) \to (\alpha \to \gamma)}\,(\Rightarrow\to)$$

3. Axiom 3: $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \dfrac{\beta \Rightarrow \beta \qquad \gamma \Rightarrow \gamma}{\beta, \beta \to \gamma \Rightarrow \gamma}\,(\to\Rightarrow)}{\alpha, \beta, \alpha \to (\beta \to \gamma) \Rightarrow \gamma}\,(\to\Rightarrow)}{\beta, \alpha \to (\beta \to \gamma) \Rightarrow \alpha \to \gamma}\,(\Rightarrow\to)}{\alpha \to (\beta \to \gamma) \Rightarrow \beta \to (\alpha \to \gamma)}\,(\Rightarrow\to)}{\Rightarrow (\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))}\,(\Rightarrow\to)$$

4. Axiom 4: $0 \to \alpha$

$$\dfrac{\dfrac{0 \Rightarrow}{0 \Rightarrow \alpha}\,(w \Rightarrow)}{\Rightarrow 0 \to \alpha}\,(\Rightarrow\to)$$

5. Axiom 5: $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \gamma \qquad \dfrac{\alpha \Rightarrow \alpha \qquad \beta \Rightarrow \beta}{\alpha \to \beta, \alpha \Rightarrow \beta}\,(\to\Rightarrow)}{\alpha \to \beta, \gamma \to \alpha, \gamma \Rightarrow \beta}\,(\to\Rightarrow)}{\alpha \to \beta, \gamma \to \alpha \Rightarrow \gamma \to \beta}\,(\Rightarrow\to)}{\alpha \to \beta \Rightarrow (\gamma \to \alpha) \to (\gamma \to \beta)}\,(\Rightarrow\to)}{\Rightarrow (\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))}\,(\Rightarrow\to)$$

6. Axiom 6: $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \lor \beta) \to \gamma))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \gamma \Rightarrow \gamma}{\alpha \to \gamma, \alpha \Rightarrow \gamma}\,(\to\Rightarrow)}{\alpha \to \gamma, \beta \to \gamma, \alpha \Rightarrow \gamma}\,(w \Rightarrow) \qquad \dfrac{\dfrac{\beta \Rightarrow \beta \qquad \gamma \Rightarrow \gamma}{\beta \to \gamma, \beta \Rightarrow \gamma}\,(\to\Rightarrow)}{\alpha \to \gamma, \beta \to \gamma, \beta \Rightarrow \gamma}\,(w \Rightarrow)}{\alpha \to \gamma, \beta \to \gamma, \alpha \lor \beta \Rightarrow \gamma}\,(\lor \Rightarrow)}{\alpha \to \gamma, \beta \to \gamma \Rightarrow (\alpha \lor \beta) \to \gamma}\,(\Rightarrow\to)}{\alpha \to \gamma \Rightarrow (\beta \to \gamma) \to ((\alpha \lor \beta) \to \gamma)}\,(\Rightarrow\to)}{\Rightarrow (\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \lor \beta) \to \gamma))}\,(\Rightarrow\to)$$

7. Axiom 7: $\alpha \to (\alpha \lor \beta)$

$$\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \Rightarrow \alpha \lor \beta}\,(\Rightarrow \lor_1)}{\Rightarrow \alpha \to (\alpha \lor \beta)}\,(\Rightarrow\to)$$

8. Axiom 8: $\beta \to (\alpha \lor \beta)$

$$\dfrac{\dfrac{\beta \Rightarrow \beta}{\beta \Rightarrow \alpha \lor \beta}\,(\Rightarrow \lor_2)}{\Rightarrow \beta \to (\alpha \lor \beta)}\,(\Rightarrow\to)$$

9. Axiom 9: $(\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \land \beta)))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \gamma \qquad \alpha \Rightarrow \alpha}{\gamma \to \alpha, \gamma \Rightarrow \alpha}\,(\to\Rightarrow)}{\gamma \to \alpha, \gamma \to \beta, \gamma \Rightarrow \alpha}\,(w \Rightarrow) \qquad \dfrac{\dfrac{\gamma \Rightarrow \gamma \qquad \beta \Rightarrow \beta}{\gamma \to \beta, \gamma \Rightarrow \beta}\,(\to\Rightarrow)}{\gamma \to \alpha, \gamma \to \beta, \gamma \Rightarrow \beta}\,(w \Rightarrow)}{\gamma \to \alpha, \gamma \to \beta, \gamma \Rightarrow \alpha \land \beta}\,(\Rightarrow \land)}{\gamma \to \alpha, \gamma \to \beta \Rightarrow \gamma \to (\alpha \land \beta)}\,(\Rightarrow\to)}{\gamma \to \alpha \Rightarrow (\gamma \to \beta) \to (\gamma \to (\alpha \land \beta))}\,(\Rightarrow\to)}{\Rightarrow (\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \land \beta)))}\,(\Rightarrow\to)$$

10. Axiom 10: $(\alpha \land \beta) \to \alpha$

$$\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \land \beta \Rightarrow \alpha}\,(\land_1 \Rightarrow)}{\Rightarrow (\alpha \land \beta) \to \alpha}\,(\Rightarrow\to)$$

11. Axiom 11: $(\alpha \land \beta) \to \beta$

$$\dfrac{\dfrac{\beta \Rightarrow \beta}{\alpha \land \beta \Rightarrow \beta}\,(\land_2 \Rightarrow)}{\Rightarrow (\alpha \land \beta) \to \beta}\,(\Rightarrow\to)$$

12. Axiom 12: $\neg\neg\alpha \to \alpha$

$$\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Rightarrow \neg\alpha, \alpha}\,(\Rightarrow \neg)}{\neg\neg\alpha \Rightarrow \alpha}\,(\neg \Rightarrow)}{\Rightarrow \neg\neg\alpha \to \alpha}\,(\Rightarrow\to)$$

The axioms are listed below for your convenience.

1. $\alpha \to (\beta \to \alpha)$
2. $(\alpha \to (\alpha \to \gamma)) \to (\alpha \to \gamma)$
3. $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$
4. $0 \to \alpha$
5. $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$
6. $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$
7. $\alpha \to (\alpha \vee \beta)$
8. $\beta \to (\alpha \vee \beta)$
9. $(\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta)))$
10. $(\alpha \wedge \beta) \to \alpha$
11. $(\alpha \wedge \beta) \to \beta$
12. $\neg\neg\alpha \to \alpha$

(b)

Solution:

If A is provable in LK_2^0, it is provable in LK^0 trivially: Any proof $\Rightarrow_{LK_2^0} A$ is also a proof in LK^0.

If A is provable in LK^0, then it can be proven with the validities of the HK axioms. Because in a) we have shown those to all be provable in LK_2^0, any proof of A in LK^0 is provable in LK_2^0. This also relies on the Modus Ponens rule in HK, but those can be simulated in LK_2^0 using the cut rule.

Thus, A is provable in LK^0 iff A is provable in LK_2^0.

(c)

Solution:

Show that the sequent $\neg\neg(\alpha \vee \neg\alpha), \alpha \vee \neg\alpha \to \neg\beta \Rightarrow \neg\neg\neg\beta$ is provable.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \Rightarrow \alpha \vee \neg\alpha}\,(\Rightarrow \vee_1) \qquad \dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\neg\alpha, \alpha \Rightarrow}\,(\neg \Rightarrow)}{\neg\alpha \Rightarrow \neg\alpha}\,(\Rightarrow \neg)}{\neg\alpha \Rightarrow \alpha \vee \neg\alpha}\,(\Rightarrow \vee_2)}{\alpha \vee \neg\alpha \Rightarrow \alpha \vee \neg\alpha}\,(\vee \Rightarrow) \qquad \dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta}{\neg\beta, \beta \Rightarrow}\,(\neg \Rightarrow)}{\neg\beta \Rightarrow \neg\beta}\,(\Rightarrow \neg)}{\neg\beta, \neg\neg\beta \Rightarrow}\,(\neg \Rightarrow)}{\dfrac{\alpha \vee \neg\alpha \to \neg\beta, \alpha \vee \neg\alpha, \neg\neg\beta \Rightarrow}{\dfrac{\alpha \vee \neg\alpha \to \neg\beta, \neg\neg\beta \Rightarrow \neg(\alpha \vee \neg\alpha)}{\neg\neg(\alpha \vee \neg\alpha), \alpha \vee \neg\alpha \to \neg\beta, \neg\neg\beta \Rightarrow}\,(\neg \Rightarrow)}\,(\Rightarrow \neg)}\,(\to\Rightarrow)}{\neg\neg(\alpha \vee \neg\alpha), \alpha \vee \neg\alpha \to \neg\beta \Rightarrow \neg\neg\neg\beta}\,(\Rightarrow \neg)
$$

(d)

Solution:

I first show that $\neg\neg\neg\gamma \Rightarrow \neg\gamma$ and $\Rightarrow \neg\neg(\alpha \vee \neg\alpha)$ are provable.

Left Derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \gamma}{\neg\gamma, \gamma \Rightarrow}\,(\neg \Rightarrow)}{\gamma \Rightarrow \neg\neg\gamma}\,(\Rightarrow \neg)}{\neg\neg\neg\gamma, \gamma \Rightarrow}\,(\neg \Rightarrow)}{\neg\neg\neg\gamma \Rightarrow \neg\gamma}\,(\Rightarrow \neg)
$$

Right Derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \Rightarrow \alpha \vee \neg\alpha}\,(\Rightarrow \vee_1)}{\neg(\alpha \vee \neg\alpha), \alpha \Rightarrow}\,(\neg \Rightarrow)}{\neg(\alpha \vee \neg\alpha) \Rightarrow \neg\alpha}\,(\Rightarrow \neg)}{\neg(\alpha \vee \neg\alpha) \Rightarrow \alpha \vee \neg\alpha}\,(\Rightarrow \vee_2)}{\neg(\alpha \vee \neg\alpha), \neg(\alpha \vee \neg\alpha) \Rightarrow}\,(\neg \Rightarrow)}{\neg(\alpha \vee \neg\alpha) \Rightarrow}\,(c \Rightarrow)}{\Rightarrow \neg\neg(\alpha \vee \neg\alpha)}\,(\Rightarrow \neg)
$$

Now I show that the provability of the premise $(\Gamma, \alpha \vee \neg\alpha \Rightarrow \neg\beta)$ implies provability of the conclusion $(\Gamma \Rightarrow \neg\beta)$.

$$
\dfrac{\dfrac{\Rightarrow \neg\neg(\alpha \vee \neg\alpha) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \alpha \vee \neg\alpha \Rightarrow \neg\beta}{\Gamma, \neg\neg\beta, \alpha \vee \neg\alpha \Rightarrow}\,(\neg \Rightarrow)}{\Gamma, \neg\neg\beta \Rightarrow \neg(\alpha \vee \neg\alpha)}\,(\Rightarrow \neg)}{\Gamma, \neg\neg(\alpha \vee \neg\alpha), \neg\neg\beta \Rightarrow}\,(\neg \Rightarrow)}{\Gamma, \neg\neg(\alpha \vee \neg\alpha) \Rightarrow \neg\neg\neg\beta}\,(\Rightarrow \neg)}{\Gamma \Rightarrow \neg\neg\neg\beta}\,\text{e-cut} \qquad \neg\neg\neg\beta \Rightarrow \neg\beta}{\Gamma \Rightarrow \neg\beta}\,\text{e-cut}
$$

We know that $\Gamma, \alpha \vee \neg\alpha \Rightarrow \neg\beta$ is provable (because of the assumption for admissibility). We have shown in the right derivation that $\Rightarrow \neg\neg(\alpha \vee \neg\alpha)$ is provable and by setting $\gamma = \beta$ we see from the left derivation that $\neg\neg\neg\beta \Rightarrow \neg\beta$ is provable. From those premises we get to the provable conclusion $\Gamma \Rightarrow \neg\beta$. Thus, the rule

$$\frac{\Gamma, \alpha \vee \neg\alpha \Rightarrow \neg\beta}{\Gamma \Rightarrow \neg\beta}$$

is admissible in $\mathbf{LJ}^0$.

2. (a)

Solution:

We can have the function $f : \mathbb{N} \times (\mathbb{N} \setminus \{0, 1, 2\}) \to \{P | P \text{ is a proof in } \mathbf{LK}\}$.

The proof will then look like the following. The idea is to define a formula A with grade k, which can be something like $p_1 \wedge \cdots \wedge p_k$, but it doesn't matter much for this proof tree. Then, on the antecedent of the conclusion, we add $m = n - 3$ propositional variables $q_1, \ldots, q_m$. After doing the e-cut with the cut formula being the formula A with grade k, we weaken m times on the left branch. Thus, we have a prooftree with grade k and height n.

$$\dfrac{\dfrac{A \Rightarrow A}{q_1, \ldots, q_m, A \Rightarrow A}\,(w \Rightarrow)\text{ m times} \qquad A \Rightarrow A}{q_1, \ldots, q_m, A \Rightarrow A}\,\text{(e-cut)}$$

(b)

Solution: With the new order $\sqsubset$, the textbook argument does not work. This is because the cut-elimination proof given in the book relies on a double induction on the grade and the height of the derivation, where the ordering is lexicographical. That is, we prioritize reducing the grade first, and the height second.

The reduction step for logical cuts runs into difficulties with this new order. Take case 4 given in the book where both of the upper sequents are lower sequents of rules for logical connectives, each of whose principal formula is the cut formula. In the example given, the proof must end as follows.

$$\dfrac{\dfrac{\Gamma \Rightarrow \beta \qquad \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \beta \wedge \gamma} \qquad \dfrac{\beta, \Delta \Rightarrow \varphi}{\beta \wedge \gamma, \Delta \Rightarrow \varphi}}{\Gamma, \Delta_{\beta \wedge \gamma} \Rightarrow \varphi}$$

In the book, when Δ contains at least one $\beta \wedge \gamma$, the proof is replaced by the following:

$$\dfrac{\Gamma \Rightarrow \beta \qquad \dfrac{\dfrac{\Gamma \Rightarrow \beta \qquad \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \beta \wedge \gamma} \qquad \beta, \Delta \Rightarrow \varphi}{\beta, \Gamma, \Delta_{\beta\wedge\gamma} \Rightarrow \varphi}\text{(e-cut)}}{\dfrac{\Gamma, \Gamma, \Delta_{\beta\wedge\gamma} \Rightarrow \varphi}{\Gamma, \Delta_{\beta\wedge\gamma} \Rightarrow \varphi}\text{some } (c \Rightarrow)}\text{(e-cut)}$$

The proof in the book relies on that the lower e-cut has a lower grade, and thus the argument works. However, with the new order $\sqsubset$ we prioritize the height n. For the lower e-cut rule, the height n actually increases. Thus, this would not work and the argument of the book does not hold anymore with this new order.

(c)

Solution:

$$\dfrac{\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \gamma}{\beta, \gamma \Rightarrow \gamma}\text{(w}\Rightarrow\text{)}}{\alpha \wedge \gamma, \beta \Rightarrow \gamma}(\wedge_2 \Rightarrow)}{\alpha \wedge \gamma \Rightarrow \beta \to \gamma}(\Rightarrow\to) \qquad \dfrac{\beta \Rightarrow \beta \qquad \dfrac{\dfrac{\beta \Rightarrow \beta}{\gamma, \beta \Rightarrow \beta}\text{(w}\Rightarrow\text{)}}{\gamma, \beta \Rightarrow \beta \vee \alpha}(\Rightarrow \vee_1)}{\beta \to \gamma, \beta, \beta \Rightarrow \beta \vee \alpha}(\to\Rightarrow)}{\alpha \wedge \gamma, \beta, \beta \Rightarrow \beta \vee \alpha}\text{e-cut (1, 10)}$$

I show the cut elimination steps.

Step 1: Case 4 applied

$$\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \gamma}{\beta, \gamma \Rightarrow \gamma}(w \Rightarrow)}{\alpha \wedge \gamma, \beta \Rightarrow \gamma}(\wedge_2 \Rightarrow) \qquad \dfrac{\dfrac{\beta \Rightarrow \beta}{\gamma, \beta \Rightarrow \beta}(w \Rightarrow)}{\gamma, \beta \Rightarrow \beta \vee \alpha}(\Rightarrow \vee_1)}{\alpha \wedge \gamma, \beta, \beta \Rightarrow \beta \vee \alpha}\text{e-cut } (0, 7)$$

Step 2: Case 3 applied

$$\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \gamma}{\gamma, \beta \Rightarrow \gamma}(w \Rightarrow) \qquad \dfrac{\dfrac{\beta \Rightarrow \beta}{\gamma, \beta \Rightarrow \beta}(w \Rightarrow)}{\gamma, \beta \Rightarrow \beta \vee \alpha}(\Rightarrow \vee_1)}{\gamma, \beta, \beta \Rightarrow \beta \vee \alpha}\text{e-cut } (0, 6)}{\alpha \wedge \gamma, \beta, \beta \Rightarrow \beta \vee \alpha}(\wedge_2 \Rightarrow)$$

Step 3: Case 3 applied

$$\dfrac{\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \gamma}{\gamma, \beta \Rightarrow \gamma}(w \Rightarrow) \qquad \dfrac{\beta \Rightarrow \beta}{\gamma, \beta \Rightarrow \beta}(w \Rightarrow)}{\gamma, \beta, \beta \Rightarrow \beta}\text{e-cut } (0, 5)}{\gamma, \beta, \beta \Rightarrow \beta \vee \alpha}(\Rightarrow \vee_1)}{\alpha \wedge \gamma, \beta, \beta \Rightarrow \beta \vee \alpha}(\wedge_2 \Rightarrow)$$

Step 4: Case 2 applied

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta}{\gamma, \beta \Rightarrow \beta}\,(w \Rightarrow) \qquad \gamma \Rightarrow \gamma}{\gamma, \beta \Rightarrow \beta}\,\text{e-cut}\,(0,4)}{\gamma, \beta, \beta \Rightarrow \beta}\,(w \Rightarrow)}{\gamma, \beta, \beta \Rightarrow \beta \vee \alpha}\,(\Rightarrow \vee_1)}{\alpha \wedge \gamma, \beta, \beta \Rightarrow \beta \vee \alpha}\,(\wedge_2 \Rightarrow)$$

Step 5: Case 1 applied

$$\dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta}{\gamma, \beta \Rightarrow \beta}\,(w \Rightarrow)}{\gamma, \beta, \beta \Rightarrow \beta}\,(w \Rightarrow)}{\gamma, \beta, \beta \Rightarrow \beta \vee \alpha}\,(\Rightarrow \vee_1)}{\alpha \wedge \gamma, \beta, \beta \Rightarrow \beta \vee \alpha}\,(\wedge_2 \Rightarrow)$$

3. (a)

Solution: The number of formulas in the language of propositional classical logic with n propositional variables when $n \geq 1$ is infinite. This can be seen easily already when $n = 1$ with a propositional variable p, where we can write as formulas $p, \neg p, \neg\neg p, \neg\neg\neg p, \ldots$.

(b)

Solution: I will show that $\sim$ is reflexive, symmetric, and transitive, and thus consequently an equivalence relation.

First, for reflexivity, we must have that for any formula α in $\mathcal{L}$ we have $\alpha \sim \alpha$. By definition of the relation $\sim$, as $h(\alpha) = h(\alpha)$ for every assignment h, we have that $\alpha \sim \alpha$. Thus, we have reflexivity.

Second, for symmetry, we must have that for all formulas α, β in $\mathcal{L}$, whenever $\alpha \sim \beta$, we also have $\beta \sim \alpha$. If we have $\alpha \sim \beta$, we thus have $h(\alpha) = h(\beta)$ for every assignment h. Because $h(\alpha) = h(\beta)$ can be rewritten as $h(\beta) = h(\alpha)$ for every assignment h (because of $=$ being an equivalence relation), we thus have $h(\beta) = h(\alpha)$ for every assignment h, and by definition of the relation $\sim$ on $\mathcal{L}$, we have $\beta \sim \alpha$. Thus, we have symmetry.

Third, we must show transitivity, meaning that for all formulas α, β, γ in $\mathcal{L}$, we have that if $\alpha \sim \beta$ and $\beta \sim \gamma$, then it follows that $\alpha \sim \gamma$. By definition of the relation $\sim$ on $\mathcal{L}$, we have $h(\alpha) = h(\beta)$ for every assignment h, and $h(\beta) = h(\gamma)$ for every assignment h. Thus, we have $h(\alpha) = h(\gamma)$ for every assignment h. Thus, by definition of the relation $\sim$ on $\mathcal{L}$, we have $\alpha \sim \gamma$.

I have shown that the relation $\sim$ on $\mathcal{L}$ is a reflexive, symmetric, and transitive binary relation. This means it is an equivalence relation.

(c)

Solution: I will show the proof tree of $(\gamma \to \alpha) \wedge (\alpha \to \gamma)$.

$$\dfrac{\dfrac{\gamma \Rightarrow \alpha}{\Rightarrow \gamma \to \alpha}(\Rightarrow\to) \qquad \dfrac{\alpha \Rightarrow \gamma}{\Rightarrow \alpha \to \gamma}(\Rightarrow\to)}{\Rightarrow (\gamma \to \alpha) \wedge (\alpha \to \gamma)}(\Rightarrow \wedge)$$

From the proof tree it is clear this proof tree is only provable in **LK** when α and γ are equivalent. By invoking the soundness and completeness theorems mentioned in Theorem 1.11 of the book, we know that this must mean that α and γ have to be semantically equivalent (i.e., $\alpha \sim \gamma$). This is exactly how the equivalence class of a formula α is defined. Thus, it holds that $[\alpha] = \{\gamma | (\gamma \to \alpha) \wedge (\alpha \to \gamma)\}$

(d)

Solution: In **LK**, for a boolean formula with n variables, there are 2^n different assignments of n variables (we are interested in the n variables because the interpolant must, as an upper bound, contain all of them). As each set of assignments of n variables can be mapped to $\{0, 1\}$, we have that there are 2^{2^n} distinct functions that represent this. Thus, the number $M_n \in \mathbb{N}$ of the upper bound for the number of interpolants of $\alpha \to \beta$ as a function of n up to logical equivalence is 2^{2^n}.

4. (a)

Solution:

1. $\alpha \to (\beta \to \alpha)$

The translation expands to:

$$\begin{aligned} T(\alpha \to (\beta \to \alpha)) &= \Box(T(\alpha) \to T(\beta \to \alpha)) \\ &= \Box(T(\alpha) \to \Box(T(\beta) \to T(\alpha))) \\ &= \Box(\Box\alpha \to \Box(\Box\beta \to \Box\alpha)) \end{aligned}$$

The proof is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Box\alpha \Rightarrow \alpha}(T)}{\Box\alpha \Rightarrow \Box\alpha}(S4)}{\Box\alpha, \Box\beta \Rightarrow \Box\alpha}(w \Rightarrow)}{\Box\alpha \Rightarrow \Box\beta \to \Box\alpha}(\Rightarrow\to)}{\Box\alpha \Rightarrow \Box(\Box\beta \to \Box\alpha)}(S4)}{\Rightarrow \Box\alpha \to \Box(\Box\beta \to \Box\alpha)}(\Rightarrow\to)}{\Rightarrow \Box(\Box\alpha \to \Box(\Box\beta \to \Box\alpha))}S4$$

2. $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$

The translation expands to:

$$\begin{aligned}
T((\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))) &= \Box\,(T(\alpha \to \beta) \to T((\gamma \to \alpha) \to (\gamma \to \beta))) \\
&= \Box\,(\Box(T(\alpha) \to T(\beta)) \to \Box\,(T(\gamma \to \alpha) \to T(\gamma \to \beta))) \\
&= \Box\,(\Box(T(\alpha) \to T(\beta)) \to \Box\,(\Box(T(\gamma) \to T(\alpha)) \to \Box(T(\gamma) \to T(\beta)))) \\
&= \Box\,(\Box(\Box\alpha \to \Box\beta) \to \Box\,(\Box(\Box\gamma \to \Box\alpha) \to \Box(\Box\gamma \to \Box\beta)))
\end{aligned}$$

The proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \gamma}{\Box\gamma \Rightarrow \gamma}\,(T)}{\Box\gamma \Rightarrow \Box\gamma}\,(S4) \quad \dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Box\alpha \Rightarrow \alpha}\,(T)}{\Box\alpha \Rightarrow \Box\alpha}\,(S4) \quad \dfrac{\dfrac{\beta \Rightarrow \beta}{\Box\beta \Rightarrow \beta}\,(T)}{\Box\beta \Rightarrow \Box\beta}\,(S4)}{\Box\alpha \to \Box\beta, \Box\alpha \Rightarrow \Box\beta}\,(\to\Rightarrow)}{\Box\alpha \to \Box\beta, \Box\gamma \to \Box\alpha, \Box\gamma \Rightarrow \Box\beta}\,(\to\Rightarrow)}{\Box\alpha \to \Box\beta, \Box\gamma \to \Box\alpha \Rightarrow \Box\gamma \to \Box\beta}\,(\Rightarrow\to)}{\Box(\Box\alpha \to \Box\beta), \Box(\Box\gamma \to \Box\alpha) \Rightarrow \Box\gamma \to \Box\beta}\,(T)}{\Box(\Box\alpha \to \Box\beta), \Box(\Box\gamma \to \Box\alpha) \Rightarrow \Box(\Box\gamma \to \Box\beta)}\,(S4)}{\Box(\Box\alpha \to \Box\beta) \Rightarrow \Box(\Box\gamma \to \Box\alpha) \to \Box(\Box\gamma \to \Box\beta)}\,(\Rightarrow\to)}{\Box(\Box\alpha \to \Box\beta) \Rightarrow \Box(\Box(\Box\gamma \to \Box\alpha) \to \Box(\Box\gamma \to \Box\beta))}\,(S4)}{\Rightarrow \Box(\Box\alpha \to \Box\beta) \to \Box(\Box(\Box\gamma \to \Box\alpha) \to \Box(\Box\gamma \to \Box\beta))}\,(\Rightarrow\to)}{\Rightarrow \Box(\Box(\Box\alpha \to \Box\beta) \to \Box(\Box(\Box\gamma \to \Box\alpha) \to \Box(\Box\gamma \to \Box\beta)))}\,(S4)$$

3. $\alpha \to (\alpha \vee \beta)$

Translation:

$$\begin{aligned}
T(\alpha \to (\alpha \vee \beta)) &= \Box(T(\alpha) \to T(\alpha \vee \beta)) \\
&= \Box(T(\alpha) \to (T(\alpha) \vee T(\beta))) \\
&= \Box(\Box\alpha \to (\Box\alpha \vee \Box\beta))
\end{aligned}$$

The prooftree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\Box\alpha \Rightarrow \alpha}\,(T)}{\Box\alpha \Rightarrow \Box\alpha}\,(S4)}{\Box\alpha \Rightarrow \Box\alpha \vee \Box\beta}\,(\Rightarrow \vee_1)}{\Rightarrow \Box\alpha \to (\Box\alpha \vee \Box\beta)}\,(\Rightarrow\to)}{\Rightarrow \Box(\Box\alpha \to (\Box\alpha \vee \Box\beta))}\,(S4)$$

4. $(\alpha \wedge \beta) \to \beta$

Translation:

$$\begin{aligned}
T((\alpha \wedge \beta) \to \beta) &= \Box(T(\alpha \wedge \beta) \to T(\beta)) \\
&= \Box((T(\alpha) \wedge T(\beta)) \to T(\beta)) \\
&= \Box((\Box\alpha \wedge \Box\beta) \to \Box\beta)
\end{aligned}$$

The prooftree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\beta \Rightarrow \beta}{\Box\beta \Rightarrow \beta}\,(T)}{\Box\beta \Rightarrow \Box\beta}\,(S4)}{\Box\alpha \wedge \Box\beta \Rightarrow \Box\beta}\,w\Rightarrow}{\Rightarrow (\Box\alpha \wedge \Box\beta) \rightarrow \Box\beta}\,(\Rightarrow\rightarrow)}{\Rightarrow \Box((\Box\alpha \wedge \Box\beta) \rightarrow \Box\beta)}\,(S4)
$$

(b)

Solution:

To show that if $T(\alpha \rightarrow \beta)$ and $T(\alpha)$ are provable in **GS4**, then $T(\beta)$ is provable in **GS4**, I will inspect the prooftree. For $T(\alpha \rightarrow \beta) = \Box(T(\alpha) \rightarrow T(\beta)) = \Box(\Box\alpha \rightarrow \Box\beta)$ the latter part of the proof can be written as:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \beta}{\Box\alpha \rightarrow \beta}\,(T)}{\Box\alpha \Rightarrow \Box\beta}\,(S4)}{\Rightarrow \Box\alpha \rightarrow \Box\beta}\,(\Rightarrow\rightarrow)}{\Rightarrow \Box(\Box\alpha \rightarrow \Box\beta)}\,(S4)
$$

From tutorial exercise 1.3, we know that if $\alpha \Rightarrow \beta$, then $\Rightarrow \alpha \rightarrow \beta$. Thus, we have that $\alpha \rightarrow \beta$ is provable.

If $T(\alpha) = \Box\alpha$ is provable in **GS4**, we can similarly look inside the prooftree and inspect what the rule(s) should be:

$$
\dfrac{\Rightarrow \alpha}{\Rightarrow \Box\alpha}\,(S4)
$$

Thus, we have also $\Rightarrow \alpha$ and therefore α is provable. With normal modus ponens, because we have that $\alpha \rightarrow \beta$ and α are both provable, we have that β is provable. We can extend the prooftree of the provable β as follows:

$$
\dfrac{\Rightarrow \beta}{\Rightarrow \Box\beta}\,(S4)
$$

We see that $\Box\beta$ is provable, which is precisely $T(\beta)$. Thus, we see that from $T(\alpha \rightarrow \beta)$ and $T(\alpha)$ being provable in **GS4**, we get that $T(\beta)$ is provable in **GS4**. Thus, Modus Ponens is preserved under translation.

(c)

Solution:

(c) Let α be a formula provable in LJ. By the standard equivalence between the sequent calculus LJ and the Hilbert system HJ (as shown in section 1.4 of the book), there exists a proof of α in HJ. We proceed by induction on the length of this derivation in HJ.

Let the derivation be a sequence of formulas $\phi_1, \ldots, \phi_n$ where $\phi_n = \alpha$. We claim that for all $k \leq n$, the sequent $\Rightarrow T(\phi_k)$ is provable in GS4.

- **Base Case:** Suppose ϕ_k is an axiom of HJ. As stated in the problem description for part (a), the translation $T(\psi)$ of every axiom ψ of HJ is provable in GS4. Thus, $\Rightarrow T(\phi_k)$ is provable.
- **Inductive Step:** Suppose ϕ_k is obtained by Modus Ponens from premises ϕ_i and $\phi_i \to \phi_k$ (with $i < k$). By the induction hypothesis, $\Rightarrow T(\phi_i)$ and $\Rightarrow T(\phi_i \to \phi_k)$ are provable in GS4. In part (b), we established that Modus Ponens is preserved under the translation T. Therefore, $\Rightarrow T(\phi_k)$ is provable in GS4.

Since α appears in the derivation, $\Rightarrow T(\alpha)$ is provable in GS4.

5. (a)

Solution:

To prove that $\vdash_c := \{(\Gamma, \alpha) | \Gamma \Rightarrow \alpha$ is provable in **LN**$\}$ is a consequence relation, we show that $\vdash_c$ satisfies the three outlined properties:

Reflexivity: If $\alpha \in \Gamma$, then when we are proving $\Gamma \Rightarrow \alpha$, from that initial sequent $\alpha \Rightarrow \alpha$ we can repeatedly apply left-weakening until we end up with $\Gamma \Rightarrow \alpha$. $\alpha \Rightarrow \alpha$ is an initial sequent and is thus provable. Thus, we have that $\Gamma \Rightarrow \alpha$ when $\alpha \in \Gamma$. Therefore, we have $\Gamma \vdash_c \alpha$.

Monotonicity: If $\Gamma \vdash_c \alpha$ and $\Gamma \subseteq \Delta$, that means we have $\Gamma \Rightarrow \alpha$ provable in **LN**. From $\Gamma \Rightarrow \alpha$, if we try to prove $\Delta \Rightarrow \alpha$, we can simply apply left-weakening on Γ until $\Delta = \Gamma$ (this is possible as $\Gamma \subseteq \Delta$). As $\Delta \Rightarrow \alpha$, we have that $\Delta \Rightarrow \alpha$. Thus, we have that $\Delta \vdash_c \alpha$.

Cut: If $\Gamma \vdash_c \alpha$ and $\Gamma \cup \{\alpha\} \vdash_c \beta$, we have that $\Gamma \Rightarrow \alpha$ and $\Gamma, \alpha \Rightarrow \beta$. We can apply the cut rule on these:

$$\dfrac{\dfrac{\Gamma \Rightarrow \alpha \qquad \alpha, \Gamma \Rightarrow \beta}{\Gamma, \Gamma \Rightarrow \beta}\,(cut)}{\Gamma \Rightarrow \beta}\,(c \Rightarrow)$$

We see that $\Gamma \Rightarrow \beta$ is provable. Thus, we have that $\Gamma \vdash_c \beta$.

We have shown that $\vdash_c := \{(\Gamma, \alpha) | \Gamma \Rightarrow \alpha$ is provable in **LN**$\}$ is a consequence relation because it satisfies reflexivity, monotonicity, and cut.

(b)

Solution:

We show that if $\Gamma \Rightarrow \Delta, \sigma, \delta$ is provable in **LN**, then $\Gamma, \Delta \Rightarrow 1 \not\twoheadrightarrow ((1 \not\twoheadrightarrow \sigma) \not\twoheadrightarrow \delta)$ is provable in **LN**.

$$\dfrac{\dfrac{\dfrac{\Rightarrow 1}{\Gamma \Rightarrow 1}\,(w \Rightarrow)}{\Gamma, \Delta \Rightarrow 1}\,(w \Rightarrow) \qquad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \sigma, \delta}{\Gamma, 1 \Rightarrow \Delta, \sigma, \delta}\,(w \Rightarrow)}{1 \not\twoheadrightarrow \sigma, \Gamma \Rightarrow \Delta, \delta}\,(\not\twoheadrightarrow\Rightarrow)}{(1 \not\twoheadrightarrow \sigma) \not\twoheadrightarrow \delta, \Gamma \Rightarrow \Delta}\,(\not\twoheadrightarrow\Rightarrow)}{\Gamma \Rightarrow \Delta, 1 \not\twoheadrightarrow ((1 \not\twoheadrightarrow \sigma) \not\twoheadrightarrow \delta)}\,(\Rightarrow\not\twoheadrightarrow)$$

$\Rightarrow 1$ is an initial sequent, and $\Gamma \Rightarrow \Delta, \sigma, \delta$ is provable in **LN**. Thus, we can conclude that $\Gamma, \Delta \Rightarrow 1 \not\twoheadrightarrow ((1 \not\twoheadrightarrow \sigma) \not\twoheadrightarrow \delta)$ is provable in **LN**.

(c)

Solution:

We show that if $\Gamma, \sigma \Rightarrow \Delta$ and $\Gamma, \delta \Rightarrow \Delta$ are provable in **LN**, then $\Gamma, 1 \not\twoheadrightarrow ((1 \not\twoheadrightarrow \sigma) \not\twoheadrightarrow \delta) \Rightarrow \Delta$ is provable in **LN**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow 1}{1 \Rightarrow 1}\,(w \Rightarrow)}{\Gamma, 1 \Rightarrow 1}\,(w \Rightarrow)}{\Gamma, 1 \Rightarrow 1, \Delta}\,(\Rightarrow w) \qquad \dfrac{\Gamma, \sigma \Rightarrow \Delta}{\Gamma, 1, \sigma \Rightarrow \Delta}\,(w \Rightarrow)}{\Gamma, 1 \Rightarrow 1 \not\twoheadrightarrow \sigma, \Delta}\,(\Rightarrow\not\twoheadrightarrow) \qquad \dfrac{\Gamma, \delta \Rightarrow \Delta}{\Gamma, \delta, 1 \Rightarrow \Delta}\,(w \Rightarrow)}{\Gamma, 1 \Rightarrow (1 \not\twoheadrightarrow \sigma) \not\twoheadrightarrow \delta, \Delta}\,(\Rightarrow\not\twoheadrightarrow)}{\Gamma, 1 \not\twoheadrightarrow ((1 \not\twoheadrightarrow \sigma) \not\twoheadrightarrow \delta) \Rightarrow \Delta}\,(\not\twoheadrightarrow\Rightarrow)$$

$\Rightarrow 1$ is an initial sequent, and $\Gamma, \sigma \Rightarrow \Delta$ and $\Gamma, \delta \Rightarrow \Delta$ are both provable in **LN**. Thus, we can conclude that $\Gamma, 1 \not\twoheadrightarrow ((1 \not\twoheadrightarrow \sigma) \not\twoheadrightarrow \delta) \Rightarrow \Delta$ is provable in **LN**.

(d)

Solution:

Suppose that a sequent $\Gamma \Rightarrow \Delta$ is provable in **LN**. We show there exists a formula σ such that for all partitions $\langle(\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2)\rangle$ of $\Gamma \Rightarrow \Delta$,

- $\Gamma_1 \Rightarrow \Delta_1 \sigma$ and $\Gamma_2, \sigma \Rightarrow \Delta_2$ are provable in **LN**, and
- $\mathrm{Var}(\sigma) \subseteq \mathrm{Var}(\Gamma_1, \Delta_1) \cap \mathrm{Var}(\Gamma_2, \Delta_2)$.

We will argue by induction following the presentation of Maehara's method in the textbook.

We can copy most of the base case from the book.

Suppose first that $\Gamma \Rightarrow \Delta$ is the initial sequent $\varphi \Rightarrow \varphi$.We need to consider the following four cases depending on to which Γ_i and Δ_j $(i, j \in \{1, 2\})$ the formula φ belongs. Thus it is necessary to find a formula φ satisfying the second condition of interpolants on variables in each of the following:

- both $\varphi \Rightarrow \varphi, \sigma$ and $\sigma \Rightarrow$ are provable,
- both $\varphi \Rightarrow \sigma$ and $\sigma \Rightarrow \varphi$ are provable,
- both $\Rightarrow \varphi, \sigma$ and $\sigma, \varphi \Rightarrow$ are provable,
- both $\Rightarrow \sigma$ and $\sigma, \varphi \Rightarrow \varphi$ are provable.

We can conservatively extend **LN** to **LN**$_0$ as done in Maehara's method and thus have as an initial sequent $0 \Rightarrow$.

For each of the cases wee see that it is enough to take $0, \varphi, \neg\varphi$, and 1 for σ, respectively.

When $\Gamma \Rightarrow \Delta$ is the initial sequent $0 \Rightarrow$, as an interpolant we can take 0 for the partition $(\langle\{0\} : \emptyset\rangle, \langle\emptyset, \emptyset\rangle)$, and 1 for the partition $(\langle\emptyset : \emptyset\rangle, \langle\{0\} : \emptyset\rangle)$, respectively.

Suppose next that $\Gamma \Rightarrow \Delta$ is not an initial sequent. Let J be the last rule applied. We need to show that $\Gamma \Rightarrow \Delta$ has an interpolant with respect to any of its partition, by assuming that upper sequents of J have interpolants with respect to any of their partition (inductive hypothesis).

Let's first go through the case where J is $(\nrightarrow\Rightarrow)$:

$$\frac{\Gamma, \alpha \Rightarrow \beta, \Delta}{\Gamma, \alpha \nrightarrow \beta \Rightarrow \Delta}\ (\nrightarrow\Rightarrow)$$

We (luckily) only have to show it for the case where the principal formula is in the first component of the partition. From now on this won't be mentioned anymore.

Take the partition $(\langle\Gamma_1, \alpha \nrightarrow \beta : \Delta_1\rangle, \langle\Gamma_2 : \Delta_2\rangle)$.

By the hypothesis of induction, with respect to a partition $(\langle\Gamma_1, \alpha : \beta, \Delta_1\rangle, \langle\Gamma_2 : \Delta_2\rangle)$ of $\Gamma, \alpha \Rightarrow \beta, \Delta$ there exists an interpolant σ such that

- both $\Gamma_1, \alpha \Rightarrow \beta, \Delta_1, \sigma$ and $\sigma, \Gamma_2 \Rightarrow \Delta_2$ are provable,
- $\mathrm{Var}(\sigma) \subseteq \mathrm{Var}(\Gamma_1, \alpha, \Delta_1, \beta) \cap \mathrm{Var}(\Gamma_2, \Delta_2)$.

Clearly, we have that $\sigma, \Gamma_2 \Rightarrow \Delta_2$ is provable.

We also have that $\Gamma_1, \alpha \nrightarrow \beta \Rightarrow \Delta_1, \sigma$ through:

$$\frac{\Gamma_1, \alpha \Rightarrow \beta, \Delta_1, \sigma}{\Gamma_1, \alpha \nrightarrow \beta \Rightarrow \Delta_1, \sigma}\ (\nrightarrow\Rightarrow)$$

And, we have the condition that $\mathrm{Var}(\sigma) \subseteq \mathrm{Var}(\Gamma_1, \alpha \nrightarrow \beta, \Delta_1) \cap \mathrm{Var}(\Gamma_2, \Delta_2)$.

Thus, we have an interpolant σ for the sequent $\Gamma \Rightarrow \Delta$ if the last rule J was $(\nrightarrow\Rightarrow)$.

Secondly, let's go through the case where J is $(\Rightarrow\nrightarrow)$:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \qquad \beta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha \nrightarrow \beta} \, (\Rightarrow\nrightarrow)$$

Take the partition $(\langle \Gamma_1 : \Delta_1, \alpha \nrightarrow \beta \rangle, \langle \Gamma_2 : \Delta_2 \rangle)$.

By the hypothesis of induction, with respect to a partition $(\langle \Gamma_1 : \Delta_1, \alpha \rangle, \langle \Gamma_2 : \Delta_2 \rangle)$ of $\Gamma \Rightarrow \Delta, \alpha$ there exists an interpolant σ such that

- both $\Gamma_1 \Rightarrow \Delta_1, \alpha, \sigma$ and $\sigma, \Gamma_2 \Rightarrow \Delta_2$ are provable,
- $\text{Var}(\sigma) \subseteq \text{Var}(\Gamma_1, \Delta_1, \alpha) \cap \text{Var}(\Gamma_2, \Delta_2)$.

Furthermore, with respect to the partition $(\langle \beta, \Gamma_1 : \Delta_1 \rangle, \langle \Gamma_2, \Delta_2 \rangle)$ of $\beta, \Gamma \Rightarrow \Delta$ there exists an interpolant δ such that

- both $\beta, \Gamma_1 \Rightarrow \Delta_1, \delta$ and $\delta, \Gamma_2 \Rightarrow \Delta_2$,
- $\text{Var}(\delta) \subseteq \text{Var}(\beta, \Gamma_1, \Delta_1) \cap \text{Var}(\Gamma_2, \Delta_2)$

We have that $\Gamma_1 \Rightarrow \Delta_1, \alpha \nrightarrow \beta, \sigma \vee \delta$ is provable through:

$$\frac{\dfrac{\Gamma_1 \Rightarrow \Delta_1, \alpha, \sigma}{\Gamma_1 \Rightarrow \Delta_1, \alpha, \sigma \vee \delta} \, (\Rightarrow \vee 1) \qquad \dfrac{\beta, \Gamma_1 \Rightarrow \Delta_1, \delta}{\beta, \Gamma_1 \Rightarrow \Delta_1, \sigma \vee \delta} \, (\Rightarrow \vee 2)}{\Gamma_1 \Rightarrow \Delta_1, \alpha \nrightarrow \beta, \sigma \vee \delta} \, (\Rightarrow\nrightarrow)$$

We have that $\Gamma_2 \Rightarrow \Delta_2$ is provable through:

$$\frac{\sigma, \Gamma_2 \Rightarrow \Delta_2 \qquad \delta, \Gamma_2 \Rightarrow \Delta_2}{\sigma \vee \delta, \Gamma_2 \Rightarrow \Delta_2} \, (\vee \Rightarrow)$$

And, we have the condition that $\text{Var}(\sigma \vee \delta) \subseteq \text{Var}(\Gamma_1, \Delta_1, \alpha \nrightarrow \beta) \cap \text{Var}(\Gamma_2, \Delta_2)$.

Thus, we have shown that $\sigma \vee \delta$ is an interpolant for the sequent $\Gamma \Rightarrow \Delta$ if the last rule J was $(\Rightarrow\nrightarrow)$. Since $\vee$ is not a connective that can be used, we can also rewrite this as $(1 \nrightarrow ((1 \nrightarrow \alpha) \nrightarrow \beta)$ as intuitively shown in **c**.

We only have the three structural rules remaining: exchange, contraction, and weakening. These hold trivially.

(e)

Solution:

$\vdash_c$ has deductive Craig interpolation if $\{\alpha\} \vdash_c \beta$ implies that there exists a formula σ such that $\{\alpha\} \vdash_c \sigma, \{\sigma\} \vdash_c \beta$, and $\text{Var}(\sigma) \subseteq \text{Var}(\alpha) \cap \text{var}(\beta)$. From **a** we have seen $\vdash_c := \{(\Gamma, \alpha) \Gamma \Rightarrow \alpha$ is provable in **LN** is a consequence relation. In **d**, we have shown that for a sequent $\Gamma \Rightarrow \Delta$ that is provable in **LN**, there exists a formula σ such that for all partitions $\langle (\Gamma_1 : \Delta_1), (\Gamma_2 : \Delta_2) \rangle$ of $\Gamma \Rightarrow \Delta$,

- $\Gamma_1 \Rightarrow \Delta_1\sigma$ and $\Gamma_2, \sigma \Rightarrow \Delta_2$ are provable in **LN**, and
- $\text{Var}(\sigma) \subseteq \text{Var}(\Gamma_1, \Delta_1) \cap \text{Var}(\Gamma_2, \Delta_2)$.

Take $\Gamma = \alpha$ and $\Delta = \beta$. Then, we see that what we have shown in **d** satisfies the deductive Craig interpolation together with what we have shown in **a**.

6. (a)

Solution:

($\Rightarrow$) proof:

$\Rightarrow \alpha \wedge \neg\beta$ is provable in $\mathbf{FL}_e$. Because $\mathbf{FL}_e$ has cut-elimination and the following rules are invertible, we know the proof must look like the following:

$$\dfrac{\Rightarrow \alpha \qquad \dfrac{\beta \Rightarrow}{\Rightarrow \neg\beta}\,(\Rightarrow \neg)}{\Rightarrow \alpha \wedge \neg\beta}\,(\Rightarrow \wedge)$$

This is the case because from $\Rightarrow \alpha \wedge \neg\beta$ the last rule that was applied must have been the $(\Rightarrow \wedge)$ rule. From that, we know that from $\Rightarrow \neg\beta$ the only rule that could have been applied must have been the $(\Rightarrow \neg)$ rule. Thus, from knowing that $\Rightarrow \alpha \wedge \neg\beta$ is provable, and knowing the above two rules are the only ones that can be applied at those steps, we know that $\Rightarrow \alpha$ and $\beta \Rightarrow$ must both be provable.

To show that from the above, it follows that $\Rightarrow \neg(\alpha \to \beta)$ is provable in $\mathbf{FL}_e$, I will similarly show the only possible valid proof:

$$\dfrac{\dfrac{\Rightarrow \alpha \qquad \beta \Rightarrow}{\alpha \to \beta \Rightarrow}\,(\to\Rightarrow)}{\Rightarrow \neg(\alpha \to \beta)}\,(\Rightarrow \neg)$$

Here, from $\Rightarrow \neg(\alpha \to \beta)$ the only possible rule one can apply is $(\Rightarrow \neg)$. From $\alpha \to \beta$ the only rule one can apply is $(\to\Rightarrow)$.

As we end up with $\Rightarrow \alpha$ and $\beta \Rightarrow$, we know those are provable (because of the prooftree of $\Rightarrow \alpha \wedge \neg\beta$), and thus $\Rightarrow \neg(\alpha \to \beta)$ is provable. Thus, if $\Rightarrow \alpha \wedge \neg\beta$ is provable, $\Rightarrow \neg(\alpha \to \beta)$ is provable.

($\Leftarrow$) proof:

$\Rightarrow \neg(\alpha \to \beta)$ is provable. From the previous proof we know that this means $\Rightarrow \alpha$ and $\beta \Rightarrow$ are both provable. We can see from the proof tree of the previous proof that the premises from $\Rightarrow \alpha \wedge \neg\beta$ are $\Rightarrow \alpha$ and $\beta \Rightarrow$. Thus, the premises are provable and $\Rightarrow \alpha \wedge \neg\beta$ is also provable. Thus, if $\Rightarrow \neg(\alpha \to \beta)$ is provable, $\Rightarrow \alpha \wedge \neg\beta$ is provable.

As a consequence, for all formulas α and β, we have that $\Rightarrow \alpha \wedge \neg\beta$ is provable in $\mathbf{FL}_e$ iff $\Rightarrow \neg(\alpha \to \beta)$ is provable in $\mathbf{FL}_e$.

(b)

Solution:

I will show the prooftree of $\neg\neg(\gamma \vee \neg\gamma)$ in $\mathbf{FL}_{ec}$. As $\mathbf{FL}_{ec}$ can only use a subset of the rules of **LJ**, if I show a proof in $\mathbf{FL}_{ec}$ it follows that it is also provable in **LJ**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma \Rightarrow \gamma}{\gamma \Rightarrow \gamma \vee \neg\gamma}\ (\Rightarrow \vee 1)}{\neg(\gamma \vee \neg\gamma), \gamma \Rightarrow}\ (\neg \Rightarrow)}{\neg(\gamma \vee \neg\gamma) \Rightarrow \neg\gamma}\ (\Rightarrow \neg)}{\neg(\gamma \vee \neg\gamma) \Rightarrow \gamma \vee \neg\gamma}\ (\Rightarrow \vee 2)}{\neg(\gamma \vee \neg\gamma), \neg(\gamma \vee \neg\gamma) \Rightarrow}\ (\neg \Rightarrow)}{\neg(\gamma \vee \neg\gamma) \Rightarrow}\ (c \Rightarrow)}{\Rightarrow \neg\neg(\gamma \vee \neg\gamma)}\ (\Rightarrow \neg)$$

(c)

Solution:

I will show that $p \vee \neg p$ is not provable in **LJ**. Then, by consequence of $\mathbf{FL}_{ec}$ using a subset of the rules, it will follow that $p \vee \neg p$ is also not provable in $\mathbf{FL}_{ec}$. There are only two possible first rules that can be applied: $(\Rightarrow \vee 1)$ and $(\Rightarrow \vee 2)$. Here are the two prooftrees:

$$\dfrac{\Rightarrow p}{\Rightarrow p \vee \neg p}\ (\Rightarrow \vee 1)$$

$$\dfrac{\Rightarrow \neg p}{\Rightarrow p \vee \neg p}\ (\Rightarrow \vee 2)$$

It can be seen that for the first prooftree there are no more possible rules that can be applied, and $\Rightarrow p$ is not an initial sequent, thus using this rule $\Rightarrow p \vee \neg p$ is not provable.

For the second prooftree, the only possible rule we can now apply is $(\Rightarrow \neg)$:

$$\dfrac{\dfrac{p \Rightarrow}{\Rightarrow \neg p}\ (\Rightarrow \neg)}{\Rightarrow p \vee \neg p}\ (\Rightarrow \vee 2)$$

In **LJ**, we know that $p \Rightarrow$ is not provable when p is a propositional variable.

IT is clear that $p \vee \neg p$ is not provable in **LJ**. Thus, $p \vee \neg p$ is not provable in $\mathbf{FL}_{ec}$ and **LJ** where p is a propositional variable.

For a formula $A = \neg\neg(\gamma \vee \neg\gamma)$ we see that $A \vee \neg A$ is provable in both $\mathbf{FL}_{ec}$ and **LJ**:

$$\dfrac{\Rightarrow \neg\neg(\gamma \vee \neg\gamma)}{\Rightarrow \neg\neg(\gamma \vee \neg\gamma) \vee \neg\neg\neg(\gamma \vee \neg\gamma)}\ (\Rightarrow \vee 1)$$

Where $\Rightarrow \neg\neg(\gamma \vee \neg\gamma)$ is provable in $\mathbf{FL}_{ec}$ and **LJ** for all formulas γ (see 6b).

(d)

Solution:

We can set $\alpha = p \vee \neg p$ and $\beta = 0$. Then,

Then we have $(p \vee \neg p) \wedge \neg 0 = (p \vee \neg p) \wedge 1 = p \vee \neg p$. From **c** we know that $p \vee \neg p$ is not provable in $\mathbf{FL}_{ec}$ and **LJ** where p is a propositional variable.

We also have $\neg((p \vee \neg p) \to 0) = \neg\neg(p \vee \neg p)$. From **b** we know that this is provable in $\mathbf{FL}_{ec}$.

Thus, there exists formulas α and β (namely, $p \vee \neg p$ and 0, respectively) such that $\Rightarrow (\alpha \wedge \neg\beta)$ is not provable in $\mathbf{FL}_{ec}$ but where $\Rightarrow \neg(\alpha \to \beta)$ is provable in $\mathbf{FL}_{ec}$.